§1. Introduction

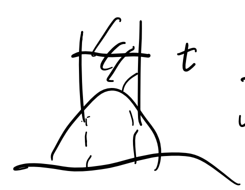
We will follow Bruinier - Funke 's paper

Goal : Define the singular theta lift from Harmonic weak Maass form of wt $= k = 1 - \frac{p}{2}$ to automorphic functions on $O(p,2)$ with certain logarithmic singularity along a divisor.

Issue: Weak Maass forms have $O(e^{cv})$ growth as $v \to \infty$ $(\tau = u + iv \in \mathbb{H})$. at least the principal part !

Recall the usual theta lifts are defined as

$$\int_{\mathcal{F}} f(\tau)\, \theta(\tau, z, \varphi)\, d\tau \qquad \mathcal{F} =$$



which may diverge because of the exp growth near the cusp (actually do diverge in our case)

The regularization :

$$\lim_{t \to \infty} \int_{\mathcal{F}_t} f(\tau)\, \theta(\tau, z)\, v^{-s}\, d\tau$$

converges for $\mathrm{Re}(s)$ large, has meromorphic continuation to nbhd of $s=0$

We take the constant term in the Laurent expansion

at $s=0$.
We will prove this and find the singularities on Z.

§2 Review on theta functions.

Dual pair $(Mp_2, O(p,2))$

Notation

$(V, (\,,\,))$ quadratic space over $\mathbb{Q}$.

$(\,,\,)$ sgn $(p,2)$

L even lattice in V, $q(x) := \frac{1}{2}(x,x) \in \mathbb{Z}$ $\forall x \in L$.

$L^{\#} :=$ dual lattice, $L^{\#}/L =$ discriminant group.

V^-, L^- same v.s. with $-(\cdot,\cdot)$

$G = SO_0(V(\mathbb{R}))$ (identity component)

$D = \{ z \in V(\mathbb{R}) \mid \dim z = 2,\ (\,,\,)|_z < 0 \} \simeq G/K$

$G' = Mp_2(\mathbb{R}) = \{ (g, \phi(\tau)) \}$

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, $\phi(\tau)^2 = c\tau + d$.

$(g_1, \phi_1(\tau))(g_2, \phi_2(\tau)) = (g_1 g_2, \phi_1(g_2\tau)\phi_2(\tau))$

Weil representation

$\omega : G \times G' \curvearrowright S(V(\mathbb{R}))$

$$\omega(g)\varphi(x) = \varphi(g^{-1}x) \qquad g \in G$$

$$\omega\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\varphi(x) = a^{\frac{p}{2}+1}\varphi(ax) \qquad a>0.$$

$$\omega\begin{pmatrix} 1 & b \\ & 1\end{pmatrix}\varphi(x) = \exp(2\pi i\, b q(x))\,\varphi(x)$$

$$\omega(S)\,\varphi(x) = \sqrt{i}^{\,2-p}\,\hat{\varphi}(-x)$$

$$S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}, \sqrt{\tau}\right), \quad \hat{\varphi}(y) = \int_{V(\mathbb{R})} \varphi(x)\, e^{2\pi i (x,y)}\, dx$$

Theta function

Let $\varphi \in S(V(\mathbb{R}))$, $h \in L^{\#}/L$

$$\theta(g', \varphi, h) = \sum_{\lambda \in L+h} \omega(g')\varphi(\lambda)$$

$$T = \left(\begin{pmatrix} 1 & 1 \\ & 1\end{pmatrix}, 1\right)$$

$$\theta(Tg', \varphi, h) = e^{2\pi i\, q(h)}\,\theta(g', \varphi, h)$$

$$\theta(Sg', \varphi, h) = \frac{\sqrt{i}^{\,2-p}}{\sqrt{|L^{\#}/L|}} \sum_{h' \in L^{\#}/L} e^{-2\pi i (h,h')}\,\theta(g', \varphi, h')$$

$$\rightsquigarrow \rho_L : Mp_2(\mathbb{Z}) \circlearrowright \mathbb{C}[L^{\#}/L]$$

vector valued theta.

$$\Theta(g', \varphi, L) := (\theta(g', \varphi, h))_{h \in L^{\#}/L}$$

$$= \sum_{h \in L^{\#}/L} \theta(g', \varphi, h)\, e_h$$

$$\Theta(\gamma g', \varphi, L) = \rho_L(\gamma)\,\Theta(g', \varphi, L) \qquad \forall \gamma \in \Gamma'$$

Let $\varphi_0(x, z) = e^{-\pi (x,x)_z}$ $\quad x \in V(\mathbb{R})$, $z \in D$

$(x, x)_z := (x_{z^\perp}, x_{z^\perp}) - (x_z, x_z)$

$x_z, x_{z^\perp}$ are given by the arthogonal projection

We have

$$\omega(k_\theta)\, \varphi_0(x, z) = e^{i\theta \cdot (\frac{p}{2}-1)} \varphi_0(x, z)$$

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

We will fix L, omit L in the variable

Define

$$\Theta(\tau, z, \varphi_0) := j(g_\tau', i)^{\frac{p}{2}-1}\, \Theta(g_\tau', \varphi, L)$$

$$= \sum_{h \in L^\#/L} \sum_{\lambda \in L+h} v \cdot e^{2\pi i (q(\lambda_{z^\perp})\tau + q(\lambda_z)\bar{\tau})}\, e_h$$

$$\left(g_\tau' = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\begin{pmatrix} v^{1/2} & \\ & v^{-1/2} \end{pmatrix}\right)$$

Satisfying

$$\Theta(\gamma\tau, z, \varphi_0) = \phi(\tau)^{p-2} \rho_L(\gamma, \phi)\, \Theta(\tau, z, \varphi_0)$$

weight $= \frac{p}{2} - 1$ in τ-variable.

Let $\Gamma \subset G$ congruence subgp fixing discriminant $L^\#/L$

Then Θ is Γ-invariant.

§3 The theta lift

Let $f \in H^{+}_{k,L^-}$ $\qquad k = 1 - \frac{p}{2}$

Recall def of weak Maass forms

- $f(\gamma\tau) = \phi(\tau)^{2k} \rho_{L^-}(\gamma, \phi) f(\tau) \quad \forall (\gamma,\phi) \in Mp_2(\mathbb{Z})$
- $\Delta_k f = 0$
- $f - P(f)$ exp. decay as $v \to \infty$

(equivalently, $\xi_k(f)$ is a cusp form)

$$f(\tau) = \sum_{h \in L^\#/L} \sum_{n+q(h) \in \mathbb{Z}} a(h,n;v)\, e(nu)\, e_h$$

$$= \sum_{h \in L^\#/L} \sum_n a^+(h,n) e(n\tau)\, e_h \qquad = f^+(\tau) + f^-(\tau)$$

$+ \cdots$

$$\boxed{P(f)(\tau) = \sum_h \sum_{n \le 0} a^+(h,n) e(n\tau)\, e_h}$$

If $f = \sum_h f_h e_h$, $\Theta(\tau, z, \varphi_0) = \sum \theta_h(\tau, z, \varphi_0) e_h$

We define

$$\Phi(z,f) = \int_{\mathcal{F}}^{reg} \sum_{h \in L^\#/L} f_h(\tau)\, \theta_h(\tau, z, \varphi_0)\, d\tau$$

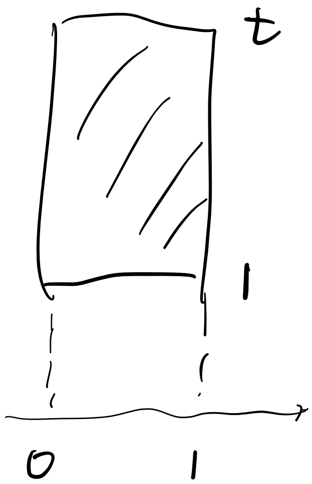
$$= C_{s=0}\left(\lim_{t\to\infty} \int_{\mathcal{F}_t} \langle f, \overline{\Theta(\tau, z, \varphi_0)} \rangle v^{-s}\, d\tau \right)$$

Proof of convergence and log singularity :

A simple reduction: $f - P(f)$ exp decay, $\mathcal{F}_1$ compact

we shall study the integral

$$\lim_{t\to\infty} \int_{\mathcal{F}_t \setminus \mathcal{F}_1} \langle P(f), \overline{\Theta(\tau, z, \varphi_0)} \rangle v^{-s} \, d\tau$$



First Integrating on u to get an expression on Fourier coefficients

$$\int_0^1 \langle P(f), \overline{\Theta(\tau, z, \varphi_0)} \rangle du \cdot v^{-s}$$

$$= \sum_{h \in L^\#/L} \sum_{\lambda \in L+h} \sum_{n \lesssim 0}$$

$$\boxed{\int_0^1 a^+(h,n) e^{2\pi i n \tau} e^{2\pi i (q(\lambda_{z^\perp})\tau + q(\lambda_z)\bar{\tau})} du} \left(v^{1-s}\right)$$

$$= a^+(h,n) \int_0^1 e(nu + q(\lambda)u) \, du$$

$$e^{-2\pi n v} \cdot e^{-2\pi q(\lambda_{z^\perp}) v} \cdot e^{2\pi q(\lambda_z) v} \quad v^{1-s}$$

$$= \delta_{n + q(\lambda) = 0} \, a^+(h,n) \exp\left(\begin{array}{l} -2\pi n v - 2\pi q(\lambda_{z^\perp}) v \\ + 2\pi q(\lambda_z) v \end{array}\right) v^{1-s}$$

$$\Rightarrow \int_0^1 \langle -- \rangle du \cdot v^{-s}$$

$$= \sum_{h \in L^\#/L} \sum_{\substack{\lambda \in L+h \\ q(\lambda) \geq 0}} a^+(h, -q(\lambda)) \exp\left(2\pi v \left(q(\lambda) + q(\lambda_z) - q(\lambda_{z^\perp}) \right)\right) v^{1-s}$$

$$= \sum_{\substack{\lambda \in L^\# \\ q(\lambda) \geq 0}} a^+(h, -q(\lambda)) \exp(4\pi q(\lambda_z) v) \; v^{1-s}$$

Back to: $\lim_{t \to \infty} \int_{F_t \setminus F_1} \langle P(f), \overline{\Theta(\tau, z, \varphi_0)} \rangle v^{-s} \frac{du\, dv}{v^2}$

$$= \int_1^\infty \sum_{\substack{\lambda \in L^\# \\ q(\lambda) \geq 0}} a^+(h, -q(\lambda)) \exp(4\pi q(\lambda_z) v) \, v^{-1-s} \, dv$$

$$= (1) \; a^+(0,0) \int_1^\infty v^{-1-s} \, dv$$

$$(2) + \int_1^\infty \sum_{\substack{\lambda \in L^\# \setminus 0 \\ q(\lambda) = 0}} a^+(h, 0) \exp(4\pi q(\lambda_z) v) \, v^{-1-s} \, dv$$

$$(3) + \int_1^\infty \sum_{\substack{\lambda \in L^\# \\ q(\lambda) > 0}} a^+(h, -q(\lambda)) \exp(4\pi q(\lambda_z) v) \, v^{-1-s} \, dv$$

First term ① $a^+(0,0)\int_1^\infty v^{-1-s}\,dv$

$$= \frac{a^+(0,0)}{s} \quad \text{if } \mathrm{Re}(s) > 0$$

Clearly $\exists$ meromorphic continuation.

Second term ② $q(\lambda) = 0 = q(\lambda_z) + q(\lambda_{z^\perp})$

$$\Rightarrow 2q(\lambda_z) = q(\lambda_z) - q(\lambda_{z^\perp})$$

$$\rightsquigarrow \int_1^\infty \sum_{\substack{\lambda \in L^\# \setminus 0 \\ q(\lambda)=0}} a^+(h,0) \exp\left(2\pi\left(q(\lambda_z) - q(\lambda_{z^\perp})\right)v\right) v^{-1-s}\,dv$$

negative definite.

$\Rightarrow$ absolutely convergent. Nothing bad.

Third term ③ This is where we get singularities

$$\int_1^\infty \sum_{n \in \mathbb{Q}_{<0}} \sum_{\substack{\lambda \in L^\# \\ q(\lambda) = -n}} a^+(h,n) \exp(4\pi q(\lambda_z) v)\, v^{-1-s}\,dv$$

— $D_x := \{z \in D \mid z \perp x\}$, $\Gamma_x = \Gamma \cap G_x$

$\Gamma_x \backslash D_x \hookrightarrow \Gamma \backslash D$, $\quad Z(h,n) := \sum_{\substack{x \in \Gamma \backslash L+h \\ q(x)=n}} Z(x)$

If $z \notin \bigcup_h \bigcup_{n<0} Z(h,-n)$

$\Rightarrow \exists \varepsilon > 0$ s.t. $q(\lambda_z) < -\varepsilon$ for all $\lambda \in L^\#$ with $q(\lambda) = -n$

$$\Rightarrow \sum_{n<0} \sum_{\substack{\lambda \in L^\# \\ q(\lambda) = -n}} \exp(4\pi q(\lambda_z) v)$$

$$\ll e^{-2\pi\varepsilon v} \cdot \sum_{n<0} \sum_{\substack{\lambda \in L^\# \\ q(\lambda) = -n}} e^{-\pi v (n - q(\lambda_z) + q(\lambda_{z^\perp}))}$$

uniformly for $y \geq 1$, locally uniformly in

To see the type of singularity we write ③ as

$$\sum_{n<0} \sum_{h \in L^\#/L} a^+(h,n) \sum_{\substack{\lambda \in h+L \\ q(\lambda) = -n}} \int_1^\infty e^{4\pi v q(\lambda_z)} v^{-s-1} \, dv$$

$$\int_1^\infty e^{4\pi v q(\lambda_z)} v^{-s-1} \, dv = \int_{-4\pi q(\lambda_z)}^\infty e^{-v} v^{-s-1} \, dv$$

$$= \Gamma(-s, -4\pi q(\lambda_z))$$

where $\Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} \, dt$

$$\Gamma(0, x) = -\gamma - \log(x) - \sum_{n \geq 1}^\infty \frac{(-x)^n}{n!\, n}$$

OR just by IBP.

$\Rightarrow$ singularity type is

$$\sum_{n<0} \sum_{h \in L^\#/L} a^+(h,n) \sum_{\substack{\lambda \in h+L \\ q(\lambda)=-n}} \left(-\log\left(-q(\lambda_z)\right) \right)$$